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Local Covariant Operator Formalism of Non-Abelian Gauge Theories

Izumi OJIMA
Research Institute for Mathematical Sciences
Kyoto University

A local and covariant operator formalism of non-Abelian gauge theories is formulated on the basis of BRS(Becchi-Rouet-Stora) invariance. By this invariance as a quantized version of local gauge invariance, criteria for physical states and observables are given so as to preserve the principles of quantum theory : elimination of negative norms, physical S-matrix unitarity, etc. In this framework of relativistic quantum field theory, the logical structure of quark confinement problem is clarified together with its criteria.

1. INTRODUCTION

All the four types of interactions ruling the nature are nowadays believed to be intermediated universally by gauge fields. Apart from (the traditional) quantum electrodynamics (QED), all these are non-Abelian gauge fields, which have been so far treated only by the path-integral method for lack of a consistent operator formalism. In spite of the powerfulness of the path-integral formalism as a calculational method, the absence in it of such standard notions as the state vector space and the Heisenberg operators obstructs us to get an insight into the general and fundamental aspects of the logical structure of the theory in a non-perturbative fashion. The understanding of these aspects seems quite necessary not only at such abstract level as the problems to assure consistency of the theory and unitarity of the physical S-matrix, etc., but also for the resolution of the outstanding problem of quark confinement, where the question "what are physically observable objects of theory ?" should be answered.

We discuss, in this article, these problems on the basis of the local and covariant Heisenberg-operator formalism of non-Abelian gauge theories obtained by T. Kugo and myself,[1,2] the essence of which is explained in §2. Its application to the quark confinement is discussed in §3 and §4, where the general consequences derived from such basic ingredients of relativistic quantum field theory (QFT) as Lorentz covariance, locality, etc., work quite effectively as technical tools, in combination with the subsidiary condition specifying physical states, the notion of observables and with the "Maxwell" equation. Throughout this formalism, a crucial role is played by a peculiar symmetry transformation, BRS(Becchi-Rouet-Stora) transformation[3], which is, roughly speaking, local gauge transformation with an unphysical scalar fermion field called Faddeev-Popov ghost[4] as its infinitesimal parameter function.

2. INDEFINITE METRIC AND SUBSIDIARY CONDITION

- Kinematical "confinement" by quartet mechanism -

The major difficulty encountered in formulating the covariant local operator formalism of gauge theories is the necessity to introduce an indefinite metric in the theory, which means the presence of negative probabilities. In order to achieve a physically meaningful interpretation, we should exclude unphysical negative norms from the "physical world" of the theory which should be defined suitably by setting up a subsidiary condition to specify a physical state. By this condition, the physical subspace $\mathcal{V}_{\text{phys}}$ consisting of physical states is picked out in the total state space $\mathcal{V}$ with indefinite metric.

Theorem 1[2] If the following three conditions are satisfied,

(0) hermiticity of the Hamiltonian H

($\Rightarrow$ pseudo-unitarity of the total S-matrix S),

(i) time invariance of $\mathcal{V}_{\text{phys}}$: $H\mathcal{V}_{\text{phys}} \subset \mathcal{V}_{\text{phys}}$

($\Rightarrow S\mathcal{V}_{\text{phys}} = S^{-1}\mathcal{V}_{\text{phys}} = \mathcal{V}_{\text{phys}}$),

(ii) positive-semidefiniteness of $\mathcal{V}_{\text{phys}}$:

$|\Phi\rangle \in \mathcal{V}_{\text{phys}} \Rightarrow \langle\Phi|\Phi\rangle \geq 0$,

then physical scattering processes are described consistently in the physical Hilbert space $H_{\text{phys}} \equiv \overline{\mathcal{V}_{\text{phys}}/\mathcal{V}_0}$ ($\mathcal{V}_0 \equiv \mathcal{V}_{\text{phys}}^{\perp} \cap \mathcal{V}_{\text{phys}}$: zero-norm suspace of $\mathcal{V}_{\text{phys}}$) in terms of the physical S-matrix S_{phys} defined by

$$S_{\text{phys}}|\hat{\Phi}\rangle \equiv \widehat{S|\Phi\rangle} \quad \text{for} \quad |\Phi\rangle \in \mathcal{V}_{\text{phys}}, \; |\hat{\Phi}\rangle \equiv |\Phi\rangle + \mathcal{V}_0 \in \mathcal{V}_{\text{phys}}/\mathcal{V}_0.$$

These H_{phys} and S_{phys} represent the ordinary state space and S-matrix in the theory without indefinite metric and the latter satisfies the usual unitarity, $S^{\dagger}_{phys}S_{phys} = S_{phys}S^{\dagger}_{phys}=1$, without the contributions from unphysical negative norms.

These conditions (0)-(ii) are shown as follows to be satisfied in our formalism with the Lagrangian density

$$\mathcal{L} \equiv \mathcal{L}_s(A,\varphi)-A^a_\mu\partial^\mu B^a+\alpha B^aB^a/2-i\partial^\mu\bar{c}^a\cdot(D_\mu c)^a, \tag{2.1a}$$

$$\mathcal{L}_s(A,\varphi) = -F^a_{\mu\nu}F^{a,\mu\nu}/4+\mathcal{L}_{matter}(\varphi,\mathcal{D}_\mu\varphi), \tag{2.1b}$$

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^a_{bc}A^b_\mu A^c_\nu,$$

$$(D_\mu c)^a = \partial_\mu c^a+gf^a_{bc}A^b_\mu c^c,$$

$$\mathcal{D}_\mu\varphi = (\partial_\mu-igA^a_\mu T^a)\varphi.$$

Here, A^a_μ is the Yang-Mills field with the group index 'a' referring to the gauge group G with the structure constant f^a_{bc}, $F^a_{\mu\nu}$ is the corresponding curvature tensor, and D_μ and $\mathcal{D}_\mu$ are the covariant derivatives associated respectively to the adjoint representation of G and the representation to which the matter field φ belongs. Terms containing B^a in (2.1a) are the gauge fixing term necessary for the canonical quantization and c^a and $\bar{c}^a$ are Faddeev-Popov (FP) ghosts whose role in retaining the gauge invariance of the physical contents of the quantum theory has been clarified by Faddeev and Popov.[4]

By the hermiticity assignment of FP ghosts, $c^\dagger = c$, $\bar{c}^\dagger = \bar{c}$, the condition (0) is satisfied: $\mathcal{L}^\dagger = \mathcal{L}$, $H^\dagger = H$. Since the system (2.1) is invariant under the BRS transformation[3] given by

$$\underline{\delta} A_\mu^a = (D_\mu c)^a, \ \underline{\delta}\varphi = igc^a T^a \varphi, \tag{2.2a}$$

$$\underline{\delta} c^a = -gf^a_{bc} c^b c^c/2, \tag{2.2b}$$

$$\underline{\delta}\bar{c}^a = iB^a, \ \underline{\delta} B^a = 0, \tag{2.2c}$$

we have the corresponding conserved Noether charge Q_B

$$Q_B \equiv \int d^3x (B^a (D_0 c)^a - \dot{B}^a c^a + igf^a_{bc} \bar{c}^a c^b c^c/2) = Q_B^\dagger, \tag{2.3}$$

satisfying the remarkable nilpotency property

$$Q_B^2 = \{Q_B, Q_B\}/2 = \underline{\delta} Q_B/2i = 0, \tag{2.4}$$

which is a consequence of the same property of $\underline{\delta}$: $\underline{\delta}^2 = 0$. Using this BRS charge Q_B, we adopt as the subsidiary condition

$$Q_B|\text{phys}\rangle = 0, \tag{2.5}$$

by which the condition (i) is automatically satisfied owing to the conservation of Q_B. Although the condition (2.5) looks quite different from the Gupta-Bleuler condition $B^{(+)}(x)|\text{phys}\rangle = 0$ in QED, it is really shown to reproduce the latter one in the Abelian cases owing to their speciality, which means that (2.5) is a natural and nontrivial extension of the Gupta-Bleuler condition to non-Abelian cases. In order to verify the condition (ii), we consider the algebra consisting of the BRS charge Q_B and the FP ghost charge Q_c defined by

$$Q_c \equiv i\int d^3x[\bar{c}^a \overleftrightarrow{\partial}_0 c^a + gf^a_{bc}\bar{c}^a A^b_0 c^c] = Q_c^\dagger, \tag{2.6}$$

$$[iQ_c, c] = c, \ [iQ_c, \bar{c}] = -\bar{c}, \tag{2.7}$$

satisfying the relations

$$[iQ_c, Q_B] = Q_B, \tag{2.8}$$

$$iQ_c|*,N\rangle = N|*,N\rangle; \quad \langle *,M|**,N\rangle \propto \delta_{M,-N} (N \in \mathbb{Z}). \tag{2.9}$$

Since all the possible representations[5,2,6] of this algebra with the properties (2.3), (2.4), (2.6), (2.8) and (2.9) are only <u>BRS-singlets</u> characterized by $Q_B|\alpha\rangle = Q_c|\alpha\rangle = 0$ without any $|*\rangle$ satisfying $Q_B|*\rangle = |\alpha\rangle$ and <u>quartets</u> consisting of pairs of BRS-doublets $\{|N\rangle, Q_B|N\rangle = |N+1\rangle; \ |-(N+1)\rangle, Q_B|-(N+1)\rangle = |-N\rangle\}$ with $\langle N|N'\rangle = \delta_{N,-N'}$, we obtain the following theorem on the assumption of asymptotic completeness:

<u>Theorem 2</u>[2] $P^{(n)} = \{Q_B, {}^\exists R^{(n)}\}$, $n \geq 1$,

where $P^{(n)}$ is the projector onto "n-unphysical-particle sector" containing n-members of quartets besides arbitrary number of BRS-singlet particles.

By this theorem, the condition (ii) is verified,

$$Q_B|f\rangle = 0 \Rightarrow \langle f|f\rangle = \langle f|\sum_{n\geq 0} P^{(n)}|f\rangle = \langle f|P^{(0)}+\{Q_B,{}^{\exists}R\}|f\rangle$$

$$= \langle P^{(0)}f|P^{(0)}f\rangle \geq 0,$$

on the inevitable assumption that all the BRS-singlet-particles have positive norms. Note that from Thm. 2 we can easily show

$$\sum_{n\geq 0} P^{(n)}\mathcal{V}_{phys} = (P^{(0)}\mathcal{V})^{\perp} \cap \mathcal{V}_{phys} = Q_B\mathcal{V} = \mathcal{V}_0, \qquad (2.10)$$

which asserts that "unphysical" particles belonging to quartets appear in $\mathcal{V}_{phys}$ but only in zero-norm combinations and that they are invisible physically. By this quartet mechanism, any member of quartets is "confined" into unphysical world, and, in the visible physical world, there remain BRS-singlets only.

In view of the quartet mechanism, quarks and gluons will be confined if they have asymptotic 1-particle states belonging to some quartets, namely if $[iQ_B,q] = ig\lambda^a c^a q/2$ and $[iQ_B,A_\mu] = D_\mu c$ have discrete poles due to the bound states in channels of c-q and A_μ-c[5].

The application of this formalism to Einstein gravity[7,8,9], vierbien formalism[7], and supergravity[10] have been extensively and successfully made.

It is interesting to note here that the physical Hilbert space $H_{phys} = \overline{\mathcal{V}_{phys}/\mathcal{V}_0}$ can be written as $\overline{\mathrm{Ker}\, Q_B/\mathrm{Im}\, Q_B}$

because of (2.5) and (2.10), which might suggest some relevance of cohomology in view of the nilpotency (2.4). The nilpotency of $\underline{\delta}$ and the resemblance of (2.2b) to the Maurer-Cartan equation also tempt us to interpret the BRS transformation as the coboundary operation in the cohomology of Lie algebra (of the infinite-dimensional group of local gauge transformations). It is, however, difficult to find the proper places for $\bar{c}$ and B to fall into. The "cohomology group" of $\underline{\delta}$ has relevance to the notion of observables discussed in the next section.

3. OBSERVABLES AND "MAXWELL" EQUATION

— Confinement vs. superselection rule —

In order to assure the consistency of the probabilistic interpretations not only in the scattering theoretical aspects so far discussed but also in the measurements of a physical quantity A, we should at least have the condition

$$\langle\Phi|A|\chi\rangle = \langle\chi|A|\Phi\rangle = 0 \quad \text{for} \quad |\Phi\rangle \in \mathcal{V}_{phys}, \ |\chi\rangle \in \mathcal{V}_0, \tag{3.1}$$

namely, no effect of zero-norms $|\chi\rangle \in \mathcal{V}_0$ should be observed through measurement of A. We call any operator A satisfying (3.1) an observable. Although this requirement seems quite modest, it leads really in this formalism to a clear result that an observable should essentially be BRS or gauge invariant:

Theorem 3[11,2]. The condition (3.1) is equivalent to

$$[Q_B, A]\mathcal{V}_{phys} = [Q_B, A^\dagger]\mathcal{V}_{phys} = 0 \tag{3.2}$$

or $\quad A\mathcal{V}_{phys} \subset \mathcal{V}_{phys}$ and $A^\dagger \mathcal{V}_{phys} \subset \mathcal{V}_{phys}$ $\quad$ (3.3)

and defines an operator $\hat{A}$ in H_{phys} by

$$\hat{A}|\hat{\Phi}\rangle = \widehat{A|\Phi\rangle} \quad \text{for } |\Phi\rangle \in \mathcal{V}_{phys}, \ |\hat{\Phi}\rangle \in \mathcal{V}_{phys}/\mathcal{V}_0. \tag{3.4}$$

Combining this theorem with the Reeh-Schlieder theorem [12,11,2] valid in a relativistic QFT which asserts $\mathcal{V} = \overline{\mathcal{F}(\mathcal{O})|0>}^{w}$ (w: weak closure) for a polynomial algebra $\mathcal{F}(\mathcal{O})$ of local operators in any finite space-time region $\mathcal{O}$, we obtain

Theorem 4[11,2] The following three conditions for a local operator $\phi \in \mathcal{F}(\mathcal{O})$ are all equivalent:

(i) ϕ is a local observable,

(ii) $[Q_B, \phi] = 0$, (3.5)

(iii) $Q_B\phi|0> = 0$. (3.6)

Now at this stage, we need dynamical information brought by the "Maxwell" equation of the Yang-Mills field[11]:

$$\partial^\mu F^a_{\mu\nu} + gJ^a_\nu = \{Q_B, (D_\nu\bar{c})^a\}, \tag{3.7}$$

where J^a_ν is the conserved Noether current of the gauge transformation of the first kind, containing the FP ghost contributions also. Combining this and Thm.4 with locality, we can prove the color singletness of local observables and of local physical states:

Theorem 5[11,2] $[\hat{Q}^a, \hat{A}] = 0$ for local observable A,

Theorem 6[11,2] $\hat{Q}^a|\hat{\Phi}> = 0$ for local physical state $|\Phi>$
$(|\Phi> \in \mathcal{V}_{phys} \cap \mathcal{F}(\mathcal{O})|0>)$,

where $\hat{Q}^a$ is an unbroken global "color" charge. So, the color confinement, $\hat{Q}^a H_{phys} = 0$, will have been achieved, for example, if the Reeh-Schlieder property holds restricted to physical subspace: $\overline{\mathcal{V}_{phys} \cap \mathcal{F}(\mathcal{O})|0\rangle}^{w} = \mathcal{V}_{phys}$. This condition does not hold automatically in contrast to the total space $\mathcal{V}$, as can be seen from the example of QED where charge superselection rule holds allowing such charged states as electrons. There are, however, some indications in favor of color confinement that charge superselection rule of QED is due to the speciality of Abelian gauge theory. Note that confinement of the global color charge leads without any gap to quark confinement, because the "behind-the-moon" argument usually placed against a confinement is not applicable to this case[11,2].

Closer analysis of the "Maxwell" equation in the next section reveals the crucial role of massless spectra in determining the fate of the color degree of freedom.

4. "MAXWELL" EQUATION AND GOLDSTONE THEOREM

— Confinement vs. Higgs phenomenon —

Rewriting the "Maxwell" equation (3.7) in the form

$$gJ^a_\mu = \partial^\nu F^a_{\mu\nu} + \{Q_B, (D_\mu \bar{c})^a\},$$

we notice that the color current gJ^a_μ consists of two conserved currents, $\partial^\nu F^a_{\mu\nu}$ and $\{Q_B, (D_\mu \bar{c})^a\}$. Since the color charge $gQ^a \equiv \int d^3x gJ^a_0$ is almost equal to the anticommutator, $N^a \equiv \int d^3x \{Q_B, (D_0\bar{c})^a\}$, of Q_B, vanishing in the physical world, up to the term $G^a \equiv \int d^3x \partial^i F^a_{0i}$ which vanishes on the local states owing to locality, the "Maxwell" equation can be naturally viewed to tend to confine its own charge. To make clear the meaning of "almost", we refer to the Goldstone theorem[13], according to which the charge G^a vanishes unless $\partial^\nu F^a_{\mu\nu}$ contains discrete massless spectrum. The same theorem, however, tells us that the charge N^a is almost always broken spontaneously owing to massless pole appearing in F.T.$\langle 0|TA_\mu B|0\rangle$ = iF.T.$\langle 0|TD_\mu c\bar{c}|0\rangle = -p_\mu/p^2$. In this case, "almost" means, "unless $\det(\mathbb{1}+u) \neq 0$", with the parameter u defined as the pole residue of $f^a_{bc}A^b_\mu \bar{c}^c$ at $p^2 = 0$.

Gathering together these sets of information, we obtain

Theorem 7[2] On the assumption $\det(\mathbb{1}+u) \neq 0$, the charge Q^a is broken spontaneously iff $\partial^\nu F^a_{\mu\nu}$ has no discrete massless spectrum.

Corollary[2] If a gauge boson becomes massive, then the charge in the corresponding channel is broken spontaneously. (Converse Higgs theorem).

So, if we do not want the color symmetry to be broken, there remain only two possibilities:

a) $\partial^\nu F^a_{\mu\nu}$ contains a massless pole ($\det(\mathbb{1}+u) \neq 0$),

or b) $u = -\mathbb{1}$.

While gluons will not be confined in the case a), the seemingly rare case b) asserts the color confinement straightforwardly: $u = -\mathbb{1} \Rightarrow G^a = 0$, $gQ^a = N^a \Rightarrow \hat{Q}^a = 0$ in H_{phys}. There may be several hurdles on the way to prove $u = -\mathbb{1}$, but once it is done, the quark confinement will have been achieved with whole consistency in this formalism of relativistic quantum theory of gauge fields.

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